Current-induced transverse spin-wave instability in thin ferromagnets: Beyond linear stability analysis

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A sufficiently large unpolarized current can cause a spin-wave instability in thin nanomagnets with asymmetric contacts. The dynamics beyond the instability is understood in the perturbative regime of small spin-wave amplitudes, as well as by numerically solving a discretized model. In the absence of an applied magnetic field, our numerical simulations reveal a hierarchy of instabilities, leading to chaotic magnetization dynamics for the largest current densities we consider.

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I. INTRODUCTION

Almost a decade ago, Slonczewski1 and Berger2 predicted that when a spin-polarized current is passed through a ferromagnet, it transfers the transverse component of its spin angular momentum to the ferromagnet. The experimental verification of the theoretical predictions followed within a few years.3–7 Since then, the so-called “spin-transfer effect” has been observed in a large number of different experiments.

In most experiments, the spin-transfer torque is studied in a ferromagnet-normal-metal-ferromagnet trilayer structure where a thick ferromagnet first polarizes the current, which then exerts a spin-transfer torque on a second thinner ferromagnet. At sufficiently large applied current densities, the spin-transfer torque then may alter the magnetization direction of the thin magnet. The observation of hysteretic magnetic switching for one current direction only was seen as a hallmark of the spin-torque effect,6 and excluded an explanation of the experiments in terms of the Faraday field.7 Almost a decade ago, Slonczewski1 and Berger2 predicted that when a spin-polarized current is passed through a ferromagnet, it transfers the transverse component of its spin angular momentum to the ferromagnet. The experimental verification of the theoretical predictions followed within a few years.3–7 Since then, the so-called “spin-transfer effect” has been observed in a large number of different experiments.

In recent work, two of the authors showed that a sufficiently large but unpolarized electrical current flowing perpendicular to a single thin ferromagnetic layer can excite spin waves in the ferromagnet.25 These spin waves have a wave vector perpendicular to the direction of current flow. The key mechanism behind the transverse spin wave instability is electron diffusion in the normal-metal contacts perpendicular to the direction of current flow; see Fig. 1. Electrons backscattered from the ferromagnet are spin polarized, the polarization direction being antiparallel to the direction of the magnetization at the location where they were reflected from the ferromagnet. When these electrons reach the ferromagnetic layer a second time, they typically do so at a different point at the normal-metal-ferromagnet interface. In the presence of a spin wave, the magnetization direction of the ferromagnet will be different at that point, and these electrons will transfer the perpendicular component of their spin to the ferromagnet, thus exerting a spin-transfer torque. The sign of this torque is to enhance the spin-wave amplitude. A similar argument can be made for electrons transmitted through the ferromagnet, but their torques tend to suppress the spin-wave amplitude. Typically, source and drain contacts are asymmetric, and a net spin-transfer torque is exerted on the ferromagnet. This torque leads to a spin wave instability for the current direction in which the effect of backscattered electrons dominates, and not for the other current direction. Experiments on nanopillars a single ferromagnetic layer found a small decrease of the device resistance above a critical current for one direction of the current and for asymmetric junctions only.29 This finding is consistent with the theoretical prediction of a dynamic spin-wave instability. A time-resolved experiment, along the lines of Refs. 8 and 9, could decide unambiguously whether the observation of Ref. 29 arises from a static or dynamic inhomogeneity.
both for single-layer and multilayer structures. In particular, Ji, Chien, and Stiles reported experimental and theoretical evidence, suggesting that for large ferromagnet thickness, ferromagnet-normal-metal junctions are unstable to the generation of nonuniform magnetization modes, but in this case, these are longitudinal modes (see also Refs. 6 and 36). See also Ref. 34 for a discussion of these experiments. Further, Stiles, Xiao, and Zangwill pointed out that transverse spin waves can be excited, even in symmetric junctions, if the spin-wave mode is not uniform in the direction of current flow. However, excitation of these modes requires a higher currents than the transverse spin waves considered here.

Our previous work, as well as the other theoretical works on this and related spin-wave instabilities, was a linear stability analysis, sufficient to predict the onset of the instability, but not to describe the spin-wave amplitude for current densities larger than the critical current density. Knowledge of the spin-wave amplitude is necessary if one wants to study, e.g., how the spin wave instability affects the resistance of the normal-metal-ferromagnet junction. It is our goal in this present work to examine in detail the dynamics of the spin wave beyond the instability. While we focus on the case of single layers, we expect that, in light of the work of Refs. 32 and 34, our qualitative findings will carry over to the case of trilayers and heterojunctions.

Although a quantitative description of how the spin-wave instability affects the resistance of the normal-metal-ferromagnet junction will be postponed to the next two sections, the sign of the effect can be determined using simple considerations. Once the current density has exceeded the critical current density for the spin-wave excitation and a spin wave has been established, the fact that the magnetization is no longer uniform reduces the amount of spin accumulation in the normal metal contacts adjacent to the ferromagnet. A reduction of the spin accumulation in the normal metal contacts causes a reduction of the sample’s resistance, see Fig. 2 for a schematic drawing. Indeed, the experiments of Ref. 29 observed a small decrease of the resistance of the nanopillar upon the onset of the spin-wave instability. The effect of a purely transverse spin-wave instability is opposite to that of a longitudinal spin-wave, which increases the resistance of the device. The reduction of the spin accumulation in the normal-metal spacer also lowers the spin-transfer torque, thus providing a mechanism to saturate the growth of the spin-wave amplitude for current densities larger than the critical current density.

For a quantitative theory of this transverse spin-wave instability, an approach that combines a full self-consistent determination of the spin-transfer torque and, at the same time, goes beyond the macrospin approximation is essential. Indeed, the macrospin approximation does not allow for non-uniform spin waves in the ferromagnet, and, whereas an externally imposed spin transfer torque would predict a similar instability, a non-self-consistent theory would be quantitatively incorrect (e.g., predict the wrong wavelength for the spin wave) because it neglects the coupling between the spin current and the spin waves in the ferromagnet.

The possibility of current-induced nonuniform modes in heterostructures has become of recent interest in the field, both for single-layer and multilayer structures. In particular, Ji, Chien, and Stiles reported experimental and theoretical evidence, suggesting that for large ferromagnet thickness, ferromagnet-normal-metal junctions are unstable to the generation of nonuniform magnetization modes, but in this case, these are longitudinal modes (see also Refs. 6 and 36). See also Ref. 34 for a discussion of these experiments. Further, Stiles, Xiao, and Zangwill pointed out that transverse spin waves can be excited, even in symmetric junctions, if the spin-wave mode is not uniform in the direction of current flow. However, excitation of these modes requires a higher currents than the transverse spin waves considered here.

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larger than the critical current density. Moreover, note that a theory of this effect needs to combine features of both the micromagnetic approach and the self-consistent treatment of the spin-transfer torque.

In Sec. II we consider current densities slightly above the critical current density. In this regime, a perturbative treatment in the spin-wave amplitude is possible. In Sec. III we then perform a detailed numerical simulation of a simplified system that allows us to probe current densities much larger than the critical current density. Whereas the observed magnetization dynamics in the presence of a large magnetic field are rather unsurprising—there is one stable energy minimum, and the magnetization precesses around the direction for which energy is minimal—in the absence of an external magnetic field we find a hierarchy of instabilities. For very high currents the system shows chaotic behavior with measurable Lyapunov exponents.

II. PERTURBATIVE CALCULATION

We consider a single ferromagnetic layer, connected to source and drain reservoirs; see Fig. 3. Between the ferromagnet and the drain reservoir there is a normal-metal spacer, as is common in nanopillar geometries. There is, however, no normal-metal spacer between the ferromagnet and the source reservoir. We use coordinates \( x, y, z \), where \( x \) is the coordinate perpendicular to the layer structure and \( y \) and \( z \) are coordinates in the plane of the layers.

Both the ferromagnet and the spacer layer have a rectangular cross section of dimensions \( W_x \times W_y \). The ferromagnet has thickness \( d \), which is small enough that the chemical potential for the conduction electrons and the direction \( \mathbf{m} \) of the magnetization of the ferromagnet do not depend on the longitudinal coordinate \( x \). The normal metal spacer has thickness \( L \). Transport through the normal metal spacer is diffusive, with conductivity \( \sigma \).

In the normal metal spacer, the charge and spin degrees of the conduction electrons are described by the equations

\[
\nabla^2 \mu_c = 0, \quad j_s = (\sigma/e) \partial_x \mu_c, \tag{1}
\]

where \( \mu_c \) and \( \mu_s \) are chemical potentials for the electron density and electron spin, respectively, \( -e \) is the electron charge, and \( l_d \) is the spin diffusion length in the normal metal spacer. Further, \( j_s \) is the charge current density and \( \sigma \) is the conductivity of the normal metal leads. The boundary conditions for \( x=L \) at the drain reservoir is

\[
\mu_c(L) = -eV, \quad \mu_s(L) = 0. \tag{2}
\]

Here the argument \( L \) refers to the \( x \) coordinate. The \( y \) and \( z \) coordinates are not written explicitly. The second boundary is the interface between the normal-metal and the ferromagnet at \( x=0 \). Since the electron dynamics occur on a time scale that is much faster than the rate of change of the magnetization direction \( \mathbf{m} \), this boundary condition can be taken, treating \( \mathbf{m} \) in the adiabatic approximation,26,27

\[
\begin{align*}
\mu_c(0) &= \frac{1}{e} [g_x \mu_c(0) + g_y \mathbf{m} \cdot \mu_s(0)], \\
\mu_s(0) &= -\frac{\hbar}{2e} [g_x \mu_c(0) + g_y \mathbf{m} \cdot \mu_s(0)] \mathbf{m} \\
&\quad + \frac{\hbar}{2e} g_z [2 \mu_s(0) \times \mathbf{m} + \hbar \mathbf{m}] \mathbf{m} \\
&\quad + \frac{\hbar}{2e} g_z [2 \mu_s(0) \times \mathbf{m} + \hbar \mathbf{m}]. \tag{3}
\end{align*}
\]

Here \( g_x = (g_x \pm g_y) / 2 \), where \( g_x \) and \( g_y \) are interface conductivities for spins aligned parallel and antiparallel to \( \mathbf{m} \), and \( g_y + ig_z \) is the complex-valued “mixing interface conductivity.” The argument “0” refers to a coordinate in the normal metal spacer, just outside the ferromagnetic layer. The charge current and the spin current parallel to \( \mathbf{m} \) are continuous at the interface. In writing down Eq. (3), we assumed that the two ferromagnet-normal-metal interfaces are identical, so that the potentials \( \mu_c \) and \( \mathbf{m} \cdot \mu_s \) drop equally over both interfaces of the ferromagnet and that the transverse electron diffusion inside the magnetic layer is negligible. The component of \( \mu_s \) perpendicular to \( \mathbf{m} \) is then zero in the ferromagnet. (It is the nonconservation of spin current perpendicular to \( \mathbf{m} \) that gives rise to the spin transfer torque.) For Co/Cu and Fe/Cr interfaces, these conductivities are tabulated; see Refs. 14 and 37. Typical values are in the range \( g_x \approx g_y < g_z \approx 10^{-14} \text{ } \Omega^{-1} \text{ } \text{m}^{-2} \). For any interface, one has the constraint \( g_y > g_z > g_x \).26

We are interested in the situation in which the magnetization is allowed to vary in the direction perpendicular to the current flow. In this case a large enough current may cause spin-wave excitations perpendicular to the direction of current flow.28 To simplify the notation, we take the limit \( \mu_x \gg l_d \). The spin and charge chemical potentials in the normal-metal spacer then have the general solution

\[
\mu_c(r) = \sum_q e^{i q_x x + i q_y y} \alpha_c(q) e^{-(q_y^2 + q_z^2)^{1/2} x} + \frac{e l x}{W_x W_y \sigma}. \tag{4}
\]
\[
\mathbf{\mu}_a(q) = \sum_q e^{i q_y y + i q_z z} e^{-\frac{1}{2} (q^2_y + q^2_z + R_a^2)^{1/2} a_a(q),}
\]

where \( q = (q_y, q_z)^T \) is a wave vector in the \( y-z \) plane. The components \( q_y \) and \( q_z \) take values \( q_y = \pi n_y/W_y, q_z = \pi n_z/W_z \) with integers \( n_y \) and \( n_z \). The Fourier expansion coefficients \( a_a(q) \) and \( a_a(q) \) are real and satisfy

\[
a_a(q) = a_a(-q), \quad a_a(q) = a_a(-q).
\]

We further define the quantities

\[
G_a(q) = (\alpha/2) (q^2_y + q^2_z)^{1/2},
\]

\[
G_a(q) = (\alpha/2) (q^2_y + q^2_z + R_a^2)^{1/2},
\]

which have the same dimension as the interface conductivities \( g_{x1}, g_1, \) and \( g_2 \). With these definitions, the boundary condition (3) at the normal-metal-ferromagnet interface becomes

\[
0 = -\frac{eI}{W_x W_z} + 2 \sum_q e^{i q_y y + i q_z z} [G_a(q) a_a(q) + g_c a_c(q) + g_m a_m(q) \cdot \mathbf{m}],
\]

\[
0 = \sum_q e^{i q_y y + i q_z z} [2G_a(q) a_a(q) + g_c a_c(q) + g_m a_m(q) \cdot \mathbf{m}] M
\]

\[
-2g_s[a_s(q) \times \mathbf{m}] \times \mathbf{m} - 2g_s(a_s(q) \times \mathbf{m})] - h g_s \mathbf{m} \times \mathbf{m}
\]

\[
- h g_s \mathbf{m}.
\]

Although Eq. (8) gives a set of linear equations for the expansion coefficients \( a_a(q) \) and \( a_a(q) \), a solution in closed form is not possible for arbitrary magnetization \( \mathbf{m}(y, z) \). Instead, we expand around the uniform equilibrium direction. Here we introduce a second coordinate system with axes labeled 1, 2, and 3, such that \( \mathbf{m} \) points along the unit vector \( \mathbf{e}_3 \) in the absence of an applied current, and write

\[
\mathbf{m} = m_1 \mathbf{e}_1 + m_2 \mathbf{e}_2 + (1 - m_1^2 - m_2^2) \mathbf{e}_3, \quad j = 1, 2, 3
\]

where \( m_j(q) = m_j(-q) \). Finally, expanding in powers of \( m_1 \) and \( m_2 \), we have solved the spin and charge chemical potentials to third order in \( m_1 \) and \( m_2 \), which parametrize the deviations from equilibrium.

In order to complete the calculation, we need to calculate the rate of change of the magnetization direction \( \mathbf{m} \) in the presence of the current \( I \). Here we use the Landau-Lifshitz-Gilbert equation,\( ^{38,39} \)

\[
\dot{\mathbf{m}} = \alpha \mathbf{m} \times \mathbf{m} + \mathbf{\tau}_{\text{ex}} + \mathbf{\tau}_{\text{an}} + \mathbf{\tau}_{\text{me}},
\]

where \( \alpha \) is the Gilbert damping coefficient, \( \mathbf{\tau}_{\text{ex}} \) is the torque arising from exchange, \( \mathbf{\tau}_{\text{an}} \) is the torque from the combined effect of magnetic anisotropy and an applied magnetic field, and \( \mathbf{\tau}_{\text{me}} \) represents the current-induced spin-transfer torque. The latter reads\( ^{40} \) as

\[
\mathbf{\tau}_{\text{me}} = \frac{\gamma}{M_d} \left[ (\mathbf{j}(0) - \mathbf{j}(-d)) \times \mathbf{m} \times \mathbf{m}
\]

\[
= -\frac{\hbar \gamma}{M d_e^2} \left[ g_1 (\mathbf{m} \times \mathbf{m} + \hbar \mathbf{m}) \times \mathbf{m} + g_2 (\mathbf{m} \times \mathbf{m} + \hbar \mathbf{m}) \right].
\]

Here the spin current \( \mathbf{j}(d) \) is taken in the source reservoir, \( M \) is the magnetization per unit volume, and \( \gamma \) is the gyromagnetic ratio. Note that the terms proportional to the time derivative \( \mathbf{m} \) have contributions from two interfaces while the contribution to the torque from the spin chemical potential has a contribution from the \( x=0 \) interface only. (All potentials are zero in the source reservoir.) The exchange torque \( \mathbf{\tau}_{\text{ex}} \) is

\[
\mathbf{\tau}_{\text{ex}} = J \gamma M \nabla^2 \mathbf{m} \times \mathbf{m},
\]

where \( J \) is the exchange constant. To linear order in \( m_1 \) and \( m_2 \), the anisotropy torque \( \mathbf{\tau}_{\text{an}} \) can be written as

\[
\mathbf{\tau}_{\text{an}} = -\frac{\gamma}{M} (k_1 m_1 \mathbf{e}_1 + k_2 m_2 \mathbf{e}_2) \times \mathbf{m},
\]

where \( k_1 \) and \( k_2 \) describe the combined effect of magnetic anisotropy and an applied magnetic field. If anisotropy dominates over the effect of a magnetic field, higher-order terms in an expansion in powers of \( m_1 \) and \( m_2 \) will be highly sample specific. Although this case can be dealt with using the methods presented below, the result of the calculation has little predictive value if those coefficients are not known independently. Therefore, we focus on the opposite limit that the anisotropy term in Eq. (14) is dominated by a magnetic field. Then higher-order terms in an expansion in powers of \( m_1 \) and \( m_2 \) are related to the first-order terms, and one has

\[
\mathbf{\tau}_{\text{an}} = (k \gamma M) \mathbf{e}_3 \times \mathbf{m},
\]

where we wrote \( k_1 = k_2 = k \). For future reference, we combine the material constants \( J \) and \( 2k = k_1 + k_2 \) into the combinations

\[
\mathbf{q}_j^2 = \frac{k}{J M^2}, \quad \mathbf{j}_j^2 = \left( \frac{2 \mathbf{e}_3}{\hbar} \right)^2 J M^2 k,
\]

which have the dimension of inverse length and current density, respectively.

We now proceed to report the result of our calculation. The lowest-order result, indicated by a superscript “(0),” is

\[
a_c^{(0)}(q) = \frac{e [g_x + 2G_s(q)]}{g_m(q) g_{\perp}} \delta_{q,0},
\]

\[
a_s^{(0)}(q) = \frac{-e_j}{g_m(q)} \mathbf{e}_3 \delta_{q,0}.
\]

Here \( j = I/W_x W_z \) is the current density and\( ^{28} \)

\[
g_m(q) = \left[ g_x + 2G_s(q) \right] g_{\perp} + 2G_s(q) - g_{\perp}. \]

Writing \( \mu_c(L) = -eV = -e(L/\sigma W_x W_z + RI) \), we conclude that the resistance \( R \) of the ferromagnetic layer is
For the zeroth-order solution, the spin potential $\mu_s$ is colinear with $m$ throughout the sample. Hence, to that order there is no current-induced torque. This is different when small deviations from the situation $m = e_2$ are taken into account to first order. One finds that the first-order corrections $a_c^{(1)}(q)$ and $a_s^{(1)}(q)$ are zero. In order to represent the first-order contributions to the transversal spin potentials $a_{s1}$ and $a_{s2}$, we use spinor notation, $a_s = (a_{s1}, a_{s2})^T$ and $m = (m_1, m_2)^T$. Then, defining

$$D(q) = [g_1 + G_s(q)]^2 + g_2^2,$$

we find

$$a_s^{(1)}(q) = -\frac{e j}{g_m(q)} m(q) + \frac{e j [G_s(q) - G_s(0)]}{g_m(0) D(q)}$$

$$\times \{ [g_1 + G_s(q)] m(q) + i g_2 \sigma_2 m(q) \}$$

$$+ \frac{h}{2 D(q)} \{ g_2 G_s(q) m(q)$$

$$+ [g_1^2 + g_1 G_s(q) + g_2^2 \sigma_2 m(q)] \},$$

where $\sigma_2$ is the second Pauli matrix. Note that the first term on the right-hand side is the response to a uniform rotation of the magnetization, while the second and third terms give the response to a nonuniform and time-dependent magnetization.

The potentials are substituted into Eq. (12) to find the current-induced torque and then into the Landau-Lifshitz-Gilbert equation (11) to find the rate of change of the magnetization. The current-induced torque has contributions proportional to the time derivative $\dot{m}$, which lead to a renormalization of the Gilbert damping parameter $\alpha$ and the gyromagnetic ratio $\gamma$. The renormalized Gilbert damping parameter $\tilde{\alpha}$ and gyromagnetic ratio $\tilde{\gamma} = \gamma/\tilde{\beta}$ depend on the transverse wave vector $q$ and read as

$$\tilde{\alpha} = \alpha + \frac{h^2}{2 M d e^2} \left(1 - \frac{G_s(q)^2}{D(q)}\right),$$

$$\tilde{\beta} = 1 + \frac{h^2}{2 M d e^2} \left(1 + \frac{G_s(q)^2}{D(q)}\right).$$

In the macrospin limit $q \rightarrow 0$, these modifications coincide with the renormalized values originally reported in Ref. 27.

Again using two-component spinor notation, the complete Landau-Lifshitz-Gilbert equation then becomes

$$(\tilde{\beta})_2 + i \sigma_2 \tilde{\alpha} m(q) = A(q)m(q),$$

with

$$A(q) = \tilde{\tau}_1^{(1)} + i \sigma_2 \left(\frac{\tilde{\tau}_2^{(1)} + \frac{h j y_j q^2 + q_f^2}{2 q f M}}{2 q f M} + \sigma_3 \frac{\gamma (k_1 - k_2)}{2 M}\right),$$

and

$$\tilde{\tau}_{\parallel}^{(1)}(q) = \frac{\hbar j y_j q^2 + q_f^2 + g_1 G_s(q)}{M d e^2} \left[ G_s(q) - g_0 \right]$$

$$\tilde{\tau}_{\perp}^{(1)}(q) = \frac{\hbar j y_j q^2 + q_f^2 + g_1 G_s(q)}{M d e^2} \left[ G_s(q) - g_0 \right].$$

In the absence of a current, any spatial modulation of the magnetization is damped. However, a sufficiently large positive current $I$ can overcome the damping, and cause a spatial modulation of $m$ to grow in time, rather than decay. (A positive current $I$ corresponds to electron flow in the negative x direction.) The instability condition is easily obtained from Eq. (23),

$$\tilde{\tau}_{\parallel}^{(1)}(q) > \frac{\hbar j y_j q^2 + q_f^2}{2 q f M} + \tilde{\tau}_{\perp}^{(1)}(q).$$

We can analyze this result in different limits. For a ferromagnetic layer with sufficiently small transverse dimensions, $W_y, W_z \approx (l_d q_f^2)^{1/3}$ if $l_d q_f \gg 1$, the instability occurs at wave vector $q = (\pi/W_y) \gamma$ or $q = (\pi/W_z) \gamma$, whichever is smallest, and the critical current follows directly from Eq. (26). For wider layers, the critical current density $j_c$ and critical wave vector $q_c$ are found as the current-density wave vector pair for which the onset of the instability condition occurs at the lowest current density.

This condition can be simplified in the limit of a very thin ferromagnetic layer, $d \rightarrow 0$, neglecting terms proportional to $g_2$ (which is numerically smaller than $g_1$), and for wave numbers $q < q_f$. We then find that the critical current follows from minimizing the relation

$$j_c(q) = \frac{h^2 g_m(q)}{2 M q f e^2} \left(1 + (1 + q^2 q_f^2)^{-1/2}\right)$$

In the limit $l_d \gg l_d q_f$, this gives

$$q_c = \left(\frac{q_f^2}{2 l_d}\right)^{1/3}, \quad j_c = \frac{h^2 g_m(q)}{2 M q f e^2}.$$

[The result for $j_c$ was reported incorrectly in Ref. 28. Note that the condition $q_c \ll q_f$, which was used to derive Eq. (27) is consistent with Eq. (28) if $l_d \gg l_d q_f$.] Note that $q_f$ increases with an applied magnetic field, so that this limit becomes relevant, even for the case of a normal metal with strong spin relaxation if the magnetic field is large enough. In the limit $l_d \ll l_d q_f$ of strong spin relaxation and weak anisotropy, one has

$$q_c = \left(\frac{4}{3}/q_f l_d\right)^{1/3}, \quad j_c = \frac{h^2 g_m(q)}{2 M q f e^2}.$$
Note that, although the applied current has a large effect on the stability of the ellipsoidal motion (precession is damped for $j < j_c$ and unstable for $j > j_c$), its effect on the precession frequency is small. To a good approximation, the precession frequency equals the ferromagnetic resonance frequency in the absence of a current.

Whereas the first-order calculation allows one to find the current density at which the spin-wave instability sets in and the angular form of the low-amplitude excitations, it does not provide information about the magnitude of the spin-wave oscillation for $j > j_c$, or about the effect of the spin-wave oscillation on the resistance of the ferromagnetic layer. This information can only be obtained from the analysis of the magnetization dynamics beyond first order in the amplitude. Such a program proceeds along the same lines as the first-order calculation shown above: the calculation of the potentials for charge and spin in the presence of a nonuniform and time-dependent current density. However, as this calculation involves higher-order contributions to the anisotropy torque $\tau_{an}$, for which the expansion constants are unknown, we find that this calculation has little predictive value. Instead, we focus on the limit in which all magnetic anisotropies arise from an applied magnetic field. In this limit, $\tau_{an}$ is known, cf. Eq. (15), and a theoretical analysis is useful.

An important simplification is that the higher-order analysis is necessary for the Fourier components $m_1(q_c)$ and $m_2(q_c)$ at the critical wave vector only. The precise value of $q_c$ is determined by the transverse boundary conditions; see the discussion following Eq. (26). For the just-above-critical current densities considered here, we can exclude a current dependent shift in $q_c$. Hence, we need to consider only a single Fourier component in our considerations below. Solving for the leading (second-order) correction to the charge potential, we find an expression that depends on the magnetization amplitude, to second order in $m_1$ and $m_2$, and on the time derivatives. Only first-order time derivatives appear, which can be eliminated using the Landau-Lifshitz-Gilbert equation (23). For the case of a large applied field, the magnetization precession is circular, and one has

$$m_1(q_c) \dot{m}_2(q_c) - m_2(q_c) \dot{m}_1(q_c) = \omega_0 r(q_c)^2,$$

where we abbreviated

$$r(q_c)^2 = m_1(q_c)^2 + m_2(q_c)^2.$$

The precession frequency $\omega_0$ given by Eq. (30). We then find

$$a^{(2)}_c(0) = \frac{2[G_s(0) - G_s(q_c)]r(q_c)^2}{D(q_c)g_m(0)^2} \times (\omega_0 g_m(0)[D(q_c) - G_s(q_c)]$$

$$\times [g_1 + G_s(q_c)] - 2 \epsilon_{ij}^c[D(q_c)$$

$$+ [G_s(0) - G_s(q_c)][g_1 + G_s(q_c)]]).$$

Solving for the leading (third-) order torque, we note that the third-order torque depends not only on the magnetization amplitudes $m_1(q_c)$ and $m_2(q_c)$, but also on their time derivative $\dot{m}_1$ and $\dot{m}_2$. The time derivatives appear to first, second, and third order in the expansion. The dependence on $\dot{m}_3^{(3)}$ is through the three-component only, which can be written as

$$\dot{m}_3^{(3)} = -m_1 \dot{m}_1^{(1)} - m_2 \dot{m}_2^{(1)}.$$

The first-order time derivatives $\dot{m}_1^{(1)}$ can be expressed in terms of $m_1$ and $m_2$ using Eq. (33) [or, in the general case, using Eq. (23)]. For the anisotropy torque $\tau_{an}$ we take the contribution from the magnetic field only. Hence

$$\tau_{an}(q_c) + \tau_{an}(q_c) = \frac{\hbar g_f}{2eq_m M} \left( q_c^2 + q_c^2 + q_c^2 \right)$$

$$\times (-i \sigma_2 m(q_c)).$$

Thus proceeding, we find that the third-order equation for the rate of change of the magnetization direction reads as

$$(\vec{B})_2 + i \sigma_2 \vec{m}(q_c)^3 = A(q_c)^3 m(q_c)^3,$$

with

$$A(q_c)^3 = -\frac{1}{2} r(q_c)^2 \left( 2 \tau^{(3)}(0)_1 + 2 \sigma_2 \tau^{(3)}(0) + r^{(3)}(2q_c)_1 \right)$$

$$- i \sigma_2 \tau^{(3)}(2q_c)_1 + 3 \omega_0 \frac{\hbar g_f q_c^2}{2eq_m \omega_0} \sigma_2$$

$$+ \hbar g_f q_c^2 \sigma_2 \frac{\omega_0}{2eq_m \omega_0},$$

and

$$\tau^{(3)}(k) = \frac{\hbar g_f}{Mde^2 [g_1 + G_s]} \left[ \left( g_1 + 2G_s(k) \left[ g_1 + 2G_s(k) \right] - g_2 \right)^2 \right]$$

$$\times \left[ \left( \frac{\epsilon_j}{g_m(0)} \left[ G_s(k) - G_s(0) \right] \right) + \left( g_1 \hbar \omega_0 \right) \right].$$

(40a)
\[ j_{c}(k) = \frac{\hbar \gamma g_{1} G_{e}(q)[G_{e}(k) - G_{e}(q)]}{M \delta^{2}[g_{1} + G_{e}(q)]} \times \left[ \frac{[g_{1} + 2G_{e}(q)][g_{1} + 2G_{e}(q)] - g_{2}^{2}}{[g_{1} + 2G_{e}(q)][g_{1} + 2G_{e}(q)] - g_{2}^{2}} \right] \left( \frac{\epsilon_{f}[g_{1} + G_{e}(0)]}{g_{m}(0)} \right) + \frac{\left( g_{1} - 2g_{1}[2G_{e}(k) + g_{1}] - g_{2}^{2}[G_{e}(q) - G_{e}(0)] \epsilon_{f} \right)}{g_{m}(0)[[g_{1} + 2G_{e}(k)][g_{1} + 2G_{e}(k)] - g_{2}^{2}} \right). \] (40b)

Solving the differential equation for \( m \), one finds that the precession amplitude for current density \( j \) slightly above the critical current density \( j_{c} \) reads as

\[ r(q_{c})^{2} = \frac{\hbar \gamma j_{f}(j_{c})}{eq_{f}Mj_{c}} \frac{\tilde{R}(q_{c})}{j_{c}(2g_{1}g_{s} + g_{s} \sigma l_{sd} - g_{2}^{2})}. \] (42)

Since \( g_{1} \gg g_{s} \), we conclude that the \( r(q_{c})^{2} \) is positive if \( j \approx j_{c} \), which excludes hysteretic behavior.

In the same limit we can also calculate the change in frequency of the spin wave, given by

\[ \frac{\omega}{\omega_{0}} = 1 + \frac{q_{c}^{2}r(q_{c})^{2}}{3(q_{c}^{2} + q_{f}^{2})^{2}}. \] (43)

Since the prefactor of the second term is much smaller than unity, \( q_{c} \leq q_{f} \) for the parameter regime of interest, we conclude that in the regime of perturbation theory, there is hardly any change from the ferromagnetic resonance frequency.

Finally, at the onset of the spin-wave instability, the resistance of the ferromagnetic layer acquires a small negative correction.

![Fig. 4](image-url)

FIG. 4. (Color online) Schematic drawing of the model solved numerically. The continuous magnet is replaced by \( N \) magnets (left), each coupled to a normal-metal wire (right). The wires are coupled via transverse diffusion (shown schematically as solid lines); the magnets are coupled via the exchange interaction (shown schematically as dashed lines).

\[ R = 1 - \frac{2(\sigma l_{sd} + 3g_{s})q_{c}^{2}r(q_{c})^{2}}{(\sigma l_{sd} + g_{s})(g_{s}^{2} + g_{s} \sigma l_{sd} - 3g_{s}^{2} - 2g_{2}^{2})}. \] (44)

(44)

In the second line we took the limits \( g_{2} \rightarrow 0, d \rightarrow 0 \), and used \( 1/l_{sd} \leq q_{c} \ll q_{f} \). This resistance decrease is anticipated on physical grounds since the nonuniform mode allows for an increased transmission of minority elections that diffuse along the transverse direction—see Fig. 2 and the corresponding discussion in Sec. I.

### III. NUMERICAL CALCULATION

The calculations in the preceding section are valid for currents close to the onset of the instability. For currents much larger than the critical current, we need to go beyond perturbation theory to obtain the dynamics. Here we numerically solve for the magnetization dynamics and its effect on the resistance of the ferromagnetic layer.

In our numerical analysis, we assume \( W_{x} \ll W_{y} \) and impose that the magnetization direction \( \mathbf{m}(y, z) \) does not depend on \( z \). The remaining two-dimensional problem is replaced by a finite number of one-dimensional problems by substituting the normal-metal spacer and the ferromagnetic layer by \( N \) normal metal channels, each attached to a magnet with magnetization direction \( \mathbf{m}(n) \), \( n = 1, \ldots, N \). In order to model a higher-dimensional structure, electrons are allowed to diffuse between the channels, whereas the \( N \) magnets interact via an exchange energy. A schematic drawing of this model is shown in Fig. 4.

In this discretized model, the potentials for charge and spin obey the equations

\[ \begin{align*}
\frac{\partial^{2}}{\partial x^{2}} \mu_{c}(n,x) + \left( \frac{N}{W_{y}} \right)^{2} \left[ \mu_{c}(n+1,x) + \mu_{c}(n-1,x) - 2\mu_{c}(n,x) \right] \\
= 0,
\end{align*} \]
\[ \partial_t^2 \mathbf{\mu}_n(x) + \left( \frac{N}{W_y} \right)^2 [\mathbf{\mu}_n(x+1) + \mathbf{\mu}_n(x-1) - 2\mathbf{\mu}_n(x)] = \frac{\mathbf{\mu}_n(x)}{t_{\text{dl}}} \]  

Equations for the boundary channels, \( n = 1 \) and \( n = N \), are obtained by setting \( \mu_{c,0}(0, x) = \mu_{c,1}(1, x) \) and \( \mu_{c,N+1}(N, x) = \mu_{c,N}(N, x) \). The general solution of Eq. (45) is of the form

\[ \mathbf{\mu}_n(x, t) = 2 \sum_{l=0}^{N-1} a_n(l) \cos[l \pi (n + 1/2)/N] e^{-q_n(l) t} + \frac{e l x}{\sigma W_s W_z} \mathbf{a}_n(l) \cos[l \pi (n + 1/2)/N] e^{-q_n(l) t}, \]

\[ \mathbf{\mu}_n(x, t) = 2 \sum_{l=0}^{N-1} a_n(l) \cos[l \pi (n + 1/2)/N] e^{-q_n(l) t}, \]

where \( q_n(l)^2 = 4(N/W_y)^2 \sin^2(l \pi/2N) \), and \( q_n(l)^2 = l_{\text{dl}}^2 + 4(N/W_y)^2 \sin^2(l \pi/2N) \).

The boundary conditions at \( x = 0 \) and \( x = N \) are given by Eq. (3).

The magnetization dynamics are given by the Landau-Lifshitz-Gilbert equation (11), with a discretized exchange torque \( \tau_{\text{ex}} \),

\[ \tau_{\text{ex}}(n) = \frac{J \gamma MN}{W_y} [\mathbf{m}(n+1) + \mathbf{m}(n-1)] \times \mathbf{m}(n). \]

For the anisotropy torque we consider two different cases: The limit of a large applied magnetic field,

\[ \tau_{\text{an}} = \frac{k \gamma}{M} \mathbf{e}_z \times \mathbf{m}(n), \]

as well as the case of no applied field, where we take a simple model for the torque arising from magnetocrystalline and shape anisotropy,

\[ \tau_{\text{an}} = -\frac{\gamma}{M} [k_1 m_1(n) \mathbf{e}_z + k_2 m_2(n) \mathbf{e}_z] \times \mathbf{m}(n). \]

The Landau-Lifshitz-Gilbert equation, together with the boundary conditions at \( x = 0 \), are sufficient to determine the 4N expansion coefficients \( a_n(l) \) and \( a_n(l, t) \), \( l = 0, \ldots, N-1 \), and the time derivative of the magnetization directions \( \mathbf{m}(n, t) \), \( n = 1, \ldots, N \). Our numerical procedure consists of first expressing \( \mathbf{m}(n) \) in terms of the potential expansion coefficients \( a_n(l) \) and \( a_n(l, t) \) using the Landau-Lifshitz-Gilbert equation, and then solving for the potential expansion coefficients using the boundary condition at \( x = 0 \).
A. Large applied magnetic field

For the numerical simulations with a magnetic field, we took values for the various parameters as follows: thickness \(d=0.2\) nm, width \(W_e=55\) nm, as is appropriate for typical nanopillar experiments,\(^7\) spin-diffusion length \(l_d=100\) nm, \(\sigma/l_d=10^{15} \Omega^{-1} \text{m}^{-2}\), \(g_1=5.5 \times 10^{14} \Omega^{-1} \text{m}^{-2}\), \(g_2=0.3 \times 10^{14} \Omega^{-1} \text{m}^{-2}\), \(g_3=g_+g_-=4.2 \times 10^{14} \Omega^{-1} \text{m}^{-2}\), \(g_4=g_+g_+=3.3 \times 10^{14} \Omega^{-1} \text{m}^{-2}\). The interface conductivities are taken from numerical calculations for a disordered Cu/Co interface;\(^4\) the conductivity \(\sigma\) and the spin relaxation length \(l_d\) are consistent with those in Cu. We further took \(\alpha=0.01\), \(\hbar g_4/M d e^2=0.0138\), \(j_f=10^{12} \text{A/m}^2\), \(q_f=10^{-3} \text{nm}^{-1}\) (as is appropriate for Co; see Ref. 41; the magnetic field corresponding to the values of \(j_f\) and \(q_f\) listed above is of a strength comparable to the intrinsic anisotropy energy). For these parameters, the width of the sample is so small that the spin-wave wave number \(q\) is set by the finite sample width, \(q=\pi/W_e\).

For current densities below \(j_c\), no spin waves are excited. Simulation runs in which the magnetization is tilted away from the easy axis \(e_3\) show damped precession toward the equilibrium magnetization direction \(m=e_3\). For current densities above \(j_c\), a spin wave with wave number \(q=\pi/W_e\) is excited. Each magnet \(n\) in our simulation \(n=1,\ldots,N\) shows circular precession around the direction of the applied magnetic field; see Fig. 5, inset. The amplitude of the oscillation increases with current, as predicted by the perturbation theory of the preceding section. The three-component of the magnetization is a constant of the motion and can be monitored to measure the amplitude. Numerical results for \(m_3\) for the magnet \(n=1\) are shown in Fig. 5 as a function of current density, together with a comparison of our numerical results with the perturbative result (41). With a large applied field, the magnetization dynamics remains regular, even for current densities much larger than \(j_c\). The effect of the spin-wave instability on the resistance of the ferromagnetic layer is shown in Fig. 6.

B. No applied magnetic field

We have also performed numerical simulations in the absence of an applied magnetic field. Here, we choose Eq. (50) for the anisotropy torque, and choose \(k_1\) and \(k_2\) such that \((k_1-k_2)/(k_1+k_2)=0.99\). This form of the anisotropy is appropriate for thin magnetic layers, in which the magnetic anisotropy is predominantly of an easy-plane type. The magnitude of the anisotropy energy is set by the parameters \(q_f\) and \(j_f\), for which we take the same values as in the previous section. All other parameters are also taken the same as in the previous section.

The magnetization dynamics without an applied magnetic field is much richer than the magnetization dynamics at a large magnetic field. The reason is the existence of two stable equilibrium directions if no external magnetic field is applied (\(m=e_3\) and \(-e_3\)). At sufficiently large current densities, the current-induced torque drives the magnetization direction between these two stable directions, leading to a variety of dynamical phases.

For the numerical parameters chosen in our simulation, we observe the following characteristic dynamical modes: For current densities \(j_e<j<2j_c\), the instability develops with the wave number \(q=\pi/W_e\). Because the magnetic anisotropy energy used for the simulation has no rotation symmetry around the three axis, the magnetization direction \(m(n)\) of each magnet \(n=1,\ldots,N\) traces out an ellipse, rather than a circle. We describe the magnetization motion is using
Poincaré sections for the polar angles \( \theta \) and \( \phi \) for the magnetization. The top right panel in Fig. 7 shows traces that are symmetric about \( \phi = \pi \), which have the functional form for \( \mathbf{m} \) as predicted by the perturbation theory in the preceding section.

For higher currents with \( 2j_c \leq j \leq 2.5j_c \), the reflection symmetry about the easy axis is spontaneously broken, resulting in asymmetric ellipses (the upper inset in Fig. 8), which for even higher current densities turn into orbits around the direction perpendicular to the easy axis (the lower inset in Fig. 8). A three-dimensional rendering of this regime is shown in Fig. 8.

For even larger currents there is a transition into nonperiodic modes that cover a significant part of phase space, as shown in Fig. 9. Whereas these modes are nonergodic for lower current densities, they eventually become ergodic and chaotic at high current densities, with Lyapunov exponents increasing with the current density \( j \) (data not shown).

In this general case, when the magnetization motion is not just simple circular precession, the spin-wave instability not only leads to a decrease of the dc resistance of the ferromagnetic layer, it also causes a fast oscillation of the resistance, as shown in the time trace in Fig. 10. The right panel in Fig. 10 shows the decrease of the dc resistance up to \( j = 2.5j_c \). (No sufficiently accurate numerical results were obtained for a larger current density.) Results for the variation of the resistance amplitude and frequency with the applied current density are shown in the left panel for current densities up to \( 4j_c \).

**IV. CONCLUSION**

We have presented a detailed study of the transverse spin-wave instability for a single ferromagnetic layer subject to a large current perpendicular to the layer. Our calculations have been in the small-amplitude regime, where perturbation theory can be used, and in the large-amplitude regime, where the magnetization dynamics can be solved numerically.

The two main signatures of the spin-wave instability are (1) the existence of the instability for one current direction only, and (2) a small reduction in the dc resistance of the ferromagnetic layer. The resistance decrease arises because the existence of a spin wave with large amplitude lowers the spin accumulation in the normal metal adjacent to the ferromagnet. A lower spin accumulation corresponds to a lower resistance (just as a high spin accumulation state of the antiparallel configuration in the standard current-perpendicular-to-plane giant magnetoresistance geometry gives a high resistance state). Both features have been seen in a recent experiment.29

An important question for a dynamical instability is whether or not it is hysteretic. Our calculation has shown that the instability studied here is not, if a large magnetic field is applied. Without an applied magnetic field, the nature of the spin wave instability depends on the precise form of the magnetic anisotropy, and both hysteretic and nonhysteretic behavior can be expected, in principle.

A noteworthy aspect of our calculation is that the spin-transfer torque is calculated self-consistently: the magnitude and direction of the spin-transfer torque depends on the spin accumulation in the normal metal, which, in turn, depends on...
the precise magnetization profile of the ferromagnet. In doing this, our work connects the circuit theory for hybrid ferromagnet-normal-metal systems, which has been used extensively to describe the magnet’s effect on spin accumulations in macrospin approximation,\textsuperscript{15} and micromagnetic simulations, which, to date, have been restricted to simplified models for the spin-transfer torque. However, our simulations should be considered a proof-of-principle. They lack the spatial resolution and sophistication that full-scale micromagnetic simulations have.

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APPENDIX: THIRD ORDER TORQUE

The perturbative calculation of Sec. II focused on the case of a large applied magnetic field because, in that case, theoretical results for the spin-wave amplitude do not depend on sample-dependent anisotropy energies. In this Appendix we outline the theory for the general case.

For the most general case, one needs a better ansatz for the intrinsic torque $\tau_m$ than Eqs. (14) and (15), as well as an expression for the current-induced torque that does not rely on rotation symmetry around the easy axis. In principle, the $\tau_m$ can be derived from the free energy $\tau_m=-(\gamma/M)\times(\partial F/\partial \mathbf{m})\times \mathbf{m}$ if there was some way to estimate the higher-order expansion coefficients. Once $\tau_m$ is known or determined empirically, the dynamics would still depend on the current-induced spin-transfer torque. We now list our general results for the second- and third-order potentials and third-order spin-transfer torque. The symbols used are defined in Sec. II of the main text. The second-order charge potential expansion coefficients for the normal-metal spacer are

$$a_c^{(2)}(q) = e^2 \sum_{q'} \left( \frac{[G_i(0) + G_i(q) - 2G_i(q')]D(q') + 2[G_i(0) - G_i(q')]G_i(q) - G_i(q')]\right) \frac{D(q')g_m(0)g_m(q)}{D(q')g_m(0)g_m(q)}$$

$$\times \left[ m_1(q')m_1(q - q') + m_2(q')m_2(q - q') \right] - \frac{2g_2[G_i(0) - G_i(q')]\left[m_1(q - q')m_1(q') + m_2(q - q')m_2(q')\right]}{[G_i(q) - G_i(q')]\left[g_1 + G_i(q')\right]} \frac{D(q')g_m(0)g_m(q)}{D(q')g_m(0)g_m(q)} + \frac{[G_i(q) - G_i(q')]\left[D(q') - G_i(q')\right]\left[g_1 + G_i(q')\right]}{D(q')g_m(q)} \frac{D(q')g_m(q)}{D(q')g_m(q)} \left[m_1(q - q')m_2(q'') - m_2(q - q')m_1(q'')\right].$$

The coefficient $a_c^{(2)}(0)$ determines how the spin-wave instability affects the resistance of the ferromagnetic layer; cf. Eq. (44) in the main text. The second-order correction to the three-component of the spin potential is given by the expansion coefficients

$$a_c^{(2)}(q) = e^2 \sum_{q'} \left( \frac{1}{2g_m(0)}[m_1(q')m_1(q - q') + m_2(q')m_2(q - q')] - \frac{1}{D(q')g_m(0)g_m(q)}\left[D(q')g_m(0)g_m(q) - 2G_i(q')\right]G_i(0) + G_i(q)$$

$$- 2G_i(q')]\left[g_1 + G_i(q')\right]g_m(q) - 2\left[g_1 + 2G_i(q')\right]\left[G_i(q) - G_i(q')\right]g_m(0)g_m(q)\right)\right)$$

$$\times \left[m_1(q')m_1(q - q') + m_2(q')m_2(q - q')\right] - \frac{2g_2G_i(0) - G_i(q')]\left[2g_1 + 2G_i(q')\right]G_i(q) - G_i(q')] - g_m(0)g_m(q)\right) \frac{D(q')g_m(0)g_m(q)}{D(q')g_m(0)g_m(q)}$$

$$\times \left[m_1(q)m_2(q - q') - m_2(q)m_1(q - q')\right] - \frac{h}{2g_m(0)} \left[2g_2G_i(q')]g_m(q) - 2\left[g_1 + 2G_i(q')\right]G_i(q) - G_i(q')]g_m(0)g_m(q)\right)\right)\right)$$

$$\times \left[m_1(q - q')m_1(q') + m_2(q - q')m_2(q')\right]$$

$$- \frac{\left[D(q') - G_i(q')\right]\left[g_1 + G_i(q')\right][g_m(q) - 2\left[g_1 + 2G_i(q')\right]G_i(q) - G_i(q')]\right] \frac{m_1(q - q')m_2(q') - m_2(q - q')m_1(q')}{2D(q')g_m(q)}.$$
The very first term describes the effect of a uniform magnetization rotation; the remaining terms are the result of a non-uniform magnetization. There are second-order corrections to the spin potential expansion coefficients $a_1$ and $a_2$ that arise from the presence of cubic terms in the anisotropy free energy. Such cubic terms cause second-order contributions to the time derivatives $\dot{m}_1$ and $\dot{m}_2$, which give a contribution to the second-order spin potentials $a^{(3)}$ in the same way as the first-order time contribution to the time derivative affects the first-order spin potentials $a^{(1)}$; see Eq. (21).

Instead of listing the third-order potentials $a^{(3)}$, we describe the corresponding current-induced torque. We specialize to the contributions that are cubic in the magnetization amplitude at wave vector $\mathbf{q}_y$. The resulting torque has terms proportional to the third-order contributions to the time derivatives of the magnetization. These terms give rise to a renormalized Gilbert damping coefficient and a renormalized gyromagnetic ratio; see Eq. (22). The remaining terms can be written as $2\tilde{\sigma}(0) + \tilde{\tau}(2\mathbf{q}_y)$, where (again using two-component spinor notation)