Conductance-peak height correlations for a Coulomb-blockaded quantum dot in a weak magnetic field

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We consider statistical correlations between the heights of conductance peaks corresponding to two different levels in a Coulomb-blockaded quantum dot. Correlations exist for two peaks at the same magnetic field if the field does not fully break time-reversal symmetry as well as for peaks at different values of a magnetic field that fully breaks time-reversal symmetry. Our results are also relevant to Coulomb-blockade conductance peak height statistics in the presence of weak spin-orbit coupling in a chaotic quantum dot.

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I. INTRODUCTION

Measurement of conductance peak heights in a Coulomb-blockaded quantum dot is one of few experimental tools to access properties of single-electron wave functions in quantum dots. Experimentally, the probability distribution of the conductance peak heights in quantum dots with an irregular shape was found to be in good agreement with predictions from random-matrix theory (RMT), both without magnetic field and in the presence of a time-reversal symmetry breaking magnetic field. According to random-matrix theory, wave functions in a chaotic quantum dot have a universal distribution, independent of details of the dot’s shape or mean free path, and wave function elements are independently Gaussian distributed real or complex numbers depending on the presence or absence of time-reversal symmetry, respectively. There are no long-range correlations within a chaotic wave function and no correlations between different wave functions.

It is known that nonuniversal correlations between different wave functions and, hence, correlations between conductance peak heights exist in both ballistic and diffusive dots. In ballistic quantum dots, such correlations are the result of wave function scarring, which causes a slow modulation of the variance \( \langle |\psi_\mu(r)|^2 \rangle \) as a function of the level index \( \mu \) and the position \( r \), although the RMT prediction for peak-height statistics remains valid for peaks at nearby energies. The scarring effect disappears in the limit of large quantum dots, and is absent in quantum dots with scatterers smaller than the Fermi wavelength. In disordered quantum dots (mean free path much smaller than dot size \( L \)), correlations between conductance peak heights are found to be of relative order \( (\Delta/E_T)\ln(L/L_d) \), where \( E_T \) is the Thouless energy of the quantum dot and \( \Delta \) the mean level spacing.

Here, we address two other mechanisms for peak-height correlations. On the one hand, we investigate correlations at a weak perpendicular magnetic field that only partially breaks time-reversal symmetry. These correlations follow from underlying correlations of wave functions, which were reported previously by Waintal, Sethna, and two of the authors. On the other hand, we also find correlations between peaks corresponding to different wave functions at two different values of a large magnetic field that fully breaks time-reversal symmetry. Unlike the two other causes for peak-height correlations, the source of correlations under investigation in this paper is universal and survives in the limit of large quantum dots. Furthermore, these correlations are not only of direct experimental relevance when Coulomb-blockade peaks are measured as a function of an external magnetic field, but our results also pertain to the case of quantum dots with weak spin-orbit coupling. We elaborate on this aspect at the end of the paper.

Our paper is organized as follows. In Sec. II, we introduce the Pandey-Mehta Hamiltonian, which is the RMT Hamiltonian appropriate for our calculations. We then proceed in Sec. III to formulate the problem in terms of orthogonal invariants of the Pandey-Mehta Hamiltonian and derive a general expression for the wave-function correlator distribution function. This result is employed to calculate the actual peak height correlator distribution function whose first moment is compared to numerical RMT simulations, both for the case of a weak magnetic field (Sec. III A) and different large magnetic fields (Sec. III B). Finally, we apply our results to correlations in presence of spin-orbit coupling and to “spectral scrambling” in Sec. IV.

II. RMT MODEL

At temperatures \( k_B T \ll \Delta \), the maximum conductance \( G_{\mu}^{\text{peak}} \) of a Coulomb-blockade conductance peak is a function of the wave function \( \psi_\mu(r) \) of the resonant state \( |\mu\rangle \) only.

\[
G_{\mu}^{\text{peak}} = \frac{e^2}{h} \frac{V \Delta}{\kappa k_B T} \left[ T_L |\psi_\mu(\bar{r}_L)|^2 + T_R |\psi_\mu(\bar{r}_R)|^2 \right] \frac{\Delta}{2 + \sqrt{\Delta^2 + T_L |\psi_\mu(\bar{r}_L)|^2 T_R |\psi_\mu(\bar{r}_R)|^2}}.
\]

Here, \( V \) is the area of the quantum dot, \( \kappa = \frac{\hbar}{\sqrt{2} m^2 e^2} \), \( \bar{r}_L \) and \( \bar{r}_R \) are the positions of the tunneling contacts connecting the dot to source and drain reservoirs (see Fig. 1), and \( T_L, T_R \ll 1 \) are the transmission probabilities of the contacts. Equation (1) is valid in the experimentally relevant range of thermally broadened conductance peaks, \( (T_L + T_R) \Delta \ll k_B T \ll \Delta \). The index \( \mu \) is defined with respect to the orbital state. In the presence of spin degeneracy, each orbital state gives rise to two conductance peaks, although these two peaks do not
The single-peak distribution \( B \) terms of the distribution of the dimensionless peak height \( s_\mu \),

\[
G_{\mu}^{\text{peak}} = g_\mu \left( \frac{e^2}{\hbar} \frac{\Delta}{k_B T} \right) \frac{T_L T_R}{T_L + T_R},
\]

and calculate the connected part of the joint conductance peak height distribution for two different levels \( \mu \) and \( \nu \),

\[
P_c(g_\mu, g_\nu) = P(g_\mu, g_\nu) - P(g_\mu)P(g_\nu).
\]

The single-peak distribution \( P(g_\mu) \) for the case of weak magnetic fields was calculated by Alhassid et al.\(^{15}\)

Within random-matrix theory, the effect of a magnetic field is described by the Pandey-Mehta Hamiltonian\(^{16}\)

\[
H(\alpha) = S + i \frac{\alpha}{\sqrt{2N}} A,
\]

where \( S \) and \( A \) are symmetric and antisymmetric \( N \times N \) matrices, respectively, with identical and independent Gaussian distributions. The parameter \( \alpha \) is proportional to the magnetic field \( B \)

\[
\alpha = e BV \frac{\sqrt{E_T}}{\hbar},
\]

where \( \gamma \) is a constant of order unity that depends on the precise geometry of the dot, for example, \( \gamma = \sqrt{\pi/2} \) for a diffusive disk of radius \( L \) \( (E_T = \hbar v_F L^2) \) and \( \gamma = \pi/\sqrt{8} \) for a ballistic disk with diffusive boundary scattering \( (E_T = \hbar v_F L),\)\(^{17-19}\)

### III. Wave Function Correlations and Peak Height Correlator

The joint distribution of eigenvectors of the Pandey-Mehta Hamiltonian (4) is determined by the orthogonal invariants \( \rho_{\mu \nu} = u_{\nu}^T \rho u_\mu \), where the superscript \( T \) denotes transposition.\(^{10}\) At fixed \( \rho_{\mu \nu} \) and for large \( N \), eigenvector components are distributed according to a multivariate Gaussian distribution with covariance matrix determined by the pair correlators

\[
\langle u^\ast_{\mu,m} u_{\nu,n} \rangle = \delta_{mn} \delta_{\mu \nu} \frac{N}{N}, \quad \langle u_{\mu,m} u_{\nu,n} \rangle = \delta_{mn} \rho_{\mu \nu} \frac{N}{N}.
\]

The distribution of the orthogonal invariants is known for the limiting cases \( |\epsilon_\mu - \epsilon_\nu| \gg \Delta \) or \( \alpha \gg 1 \), when their distribution is Gaussian with zero mean and with variance\(^{10}\)

\[
\langle |\rho_{\mu \nu}|^2 \rangle = \frac{2\alpha^2(1 + \delta_{\mu \nu})}{4\alpha^4 + \pi^2(\epsilon_\mu - \epsilon_\nu)^2 / \Delta^2}.
\]

Furthermore, if \( |\epsilon_\mu - \epsilon_\nu| \gg \Delta \) or \( \alpha \gg 1 \), \( |\rho_{\mu \nu}|^2 \) and \( |\rho_{\nu \mu}|^2 \) are statistically independent. Equation (7) is also valid in the limit \( \alpha \ll 1 \) if an additional average over the energy levels \( \epsilon_\mu - \epsilon_\nu \) is taken. No analytical results are known for the distribution of the orthogonal invariants \( \rho_{\mu \nu} \) when \( \mu \neq \nu \), \( \alpha \) is of order unity, and \( |\epsilon_\mu - \epsilon_\nu| \ll \Delta \).

Using the correspondence between the eigenvectors of the Pandey-Mehta Hamiltonian and the wave functions in the quantum dot, we identify \( \psi_\mu(\tilde{r}_R) \) with \( u_{\mu,1} \) and \( \psi_\nu(\tilde{r}_R) \) with \( u_{\nu,2} \). We are interested in the joint distribution of the wave functions corresponding to the levels \( \mu \) and \( \nu \) and abbreviate \( x_1 = N|\nu_{\mu,1}|^2 \), \( x_2 = N|\nu_{\mu,2}|^2 \), \( y_1 = N|\nu_{\nu,1}|^2 \), and \( y_2 = N|\nu_{\nu,2}|^2 \). To leading order in \( \rho_{\mu \nu} \), the connected part of the average of the form \( \langle \langle x_1 x_2 y_1 y_2 \rangle \rangle_c \) \( = \langle x_1 x_2 y_1 y_2 \rangle - \langle x_1 x_2 \rangle \times \langle y_1 y_2 \rangle \) can be calculated with the help of Wick’s theorem and Eq. (6),

\[
\langle \langle x_1 x_2 y_1 y_2 \rangle \rangle_c = \langle |\rho_{\mu \nu}|^2 \rangle \langle k^2 m^2 (x_1^{-1} x_2^{-1})(y_1^{-1} y_2^{-1}) \rangle
+ \hat{n}^2 \langle (x_1^{-1} x_2^{-1})(y_1^{-1} y_2^{-1}) \rangle,
\]

In the regimes \( |\epsilon_\mu - \epsilon_\nu| \gg \Delta \) or \( \alpha \gg 1 \), this relation allows us to express the connected part of the joint distribution function

\[
P_c(x_1, x_2; y_1, y_2) = P(x_1, x_2; y_1, y_2) - P(x_1, x_2)P(y_1, y_2)
\]

in terms of the distribution functions \( P(x_1, x_2) \) and \( P(y_1, y_2) \) for elements of a single eigenvector

\[
P_c(x_1, x_2; y_1, y_2) = \frac{2\alpha^2}{4\alpha^4 + \pi^2(\epsilon_\mu - \epsilon_\nu)^2 / \Delta^2}
\]

\[
\times \sum_{j=1}^2 D_x P(x_j, x_2) D_y P(y_1, y_j),
\]

where \( D_x = \partial_{x} \partial_{x} \). The single-wave-function distribution \( P(x_1, x_2) \) for the Pandey-Mehta Hamiltonian was calculated by Fal’ko and Efetov.\(^{20}\)

### A. Weak magnetic field

Using Eq. (8), the calculation of the peak-height correlation function \( P_c(g_\mu, g_\nu) \) becomes a matter of quadrature. Closed-form results can be obtained for the case \( \alpha \gg 1 \):

\[
P_c(g_\mu, g_\nu) = \frac{1}{2\alpha^2} (1 - g_{\mu \nu})(1 - g_{\nu \mu})
\]

for highly asymmetric tunneling contacts \( T_L \ll T_R \), whereas for symmetric contacts \( T_L = T_R \)

\[
P_c(g_\mu, g_\nu) = \frac{1}{16\alpha} W(g_\mu) W(g_\nu),
\]

\[
\text{PHYSICAL REVIEW B 68, 035323 (2003)}
\]
The degree of correlation is well characterized by the first moment of \( P_c(g_{\mu}, g_{\nu}) \).

\[
C_{\mu\nu} = \langle g_{\mu} g_{\nu} \rangle - \langle g_{\mu} \rangle \langle g_{\nu} \rangle. \tag{12}
\]

In the regime \( \alpha \gg 1 \) we find from Eqs. (9) and (10)

\[
C_{\mu\nu} = \frac{1}{2 \alpha^2} \text{ if } T_L \ll T_R, \tag{13a}
\]

\[
C_{\mu\nu} = \frac{1}{9 \alpha^2} \text{ if } T_L = T_R. \tag{13b}
\]

For very weak magnetic fields \( \alpha \ll 1 \) evaluation of the correlator \( P_c \) requires knowledge of the \( |\mu - \nu| \)-level spacing distribution functions for the Gaussian orthogonal ensemble of random-matrix theory. Although this solution to this problem is known in the form of a product of eigenvalues of a certain integral equation,\(^{21}\) no closed-form expressions exist to the best of our knowledge. Moreover, if the energy levels \( \mu \) and \( \nu \) are nearest neighbors, small-\( \alpha \) perturbation theory fails for small level separations. [Upon averaging over energy, Eq. (7) gives a logarithmic divergence.] This problem can be circumvented, noting that the orthogonal invariants \( \rho_{\mu\nu} \), the only source of correlations if \( \alpha \ll 1 \), are completely determined by properties of the two energy levels under consideration. Therefore, the peak-height correlations may be calculated using a \( 2 \times 2 \) Hamiltonian instead of the full \( N \times N \) random matrix. In the eigenvector basis of the Pandey-Mehta Hamiltonian (4) at \( \alpha = 0 \), the appropriate two-level Hamiltonian reads

\[
\mathcal{H} = \begin{pmatrix}
\varepsilon_{\mu} & \frac{i \alpha A_{\mu\nu} / \sqrt{2N}}{i \alpha A_{\nu\mu} / \sqrt{2N}} \\
\frac{i \alpha A_{\nu\mu} / \sqrt{2N}}{i \alpha A_{\mu\nu} / \sqrt{2N}} & \varepsilon_{\nu}
\end{pmatrix},
\tag{14}
\]

where \( \varepsilon_{\mu} \) and \( \varepsilon_{\nu} \) are the two energy levels at \( \alpha = 0 \), and \( A_{\mu\nu} = -A_{\nu\mu} \) is the corresponding matrix element of the perturbing matrix \( A \), see Eq. (4). Solving for the eigenvectors of \( \mathcal{H} \) and calculating the distribution of \( \rho_{\mu\nu} \) exactly, we were able to compute the small-\( \alpha \) behavior of the correlator \( C_{\mu\nu} \) for \( \nu = \mu + 1 \),

\[
C_{\mu\nu} \approx \frac{\alpha^2}{2 \pi} \ln \frac{4 \pi}{\alpha^2 e^2} \text{ if } T_L \ll T_R, \tag{15a}
\]

\[
C_{\mu\nu} \approx \frac{3 \alpha^2}{16 \pi} \ln \frac{0.569}{\alpha^2} \text{ if } T_L = T_R. \tag{15b}
\]

The numerical coefficients inside the logarithms in Eqs. (15) were obtained by making use of the Wigner surmise \( P(s) = (\pi s/2) \exp(-\pi s^2/4) \) as a numerical approximation to the distribution of nearest-neighbor spacings

\[
s = |\varepsilon_{\mu+1} - \varepsilon_{\mu}|/\Delta. \tag{21}
\]

A comparison of our results with the result of numerical diagonalizations of the Pandey-Mehta Hamiltonian is shown in Fig. 2 for the case of symmetric tunneling contacts. We used random matrices of sizes \( N = 100, 200, \) and 400 and extrapolated to \( N \to \infty \) to eliminate finite-\( N \) effects. Note that throughout the magnetic-field range of interest, correlations between peaks are positive: small peaks are more likely to be surrounded by small peaks and large peaks attract more large peaks.

**B. Different large magnetic fields**

We now turn to correlations between conductance peaks at different large values of the magnetic field. In particular, we are interested in the connected part of the joint distribution \( P(g_{\mu}, g_{\nu}) \) where \( g_{\mu} \) is a (dimensionless) peak height at magnetic-field strength \( \alpha \) while \( g_{\nu} \) is a peak height of a different orbital state at a different magnetic-field strength \( \alpha' \). (Magnetic-field autocorrelations for the same peak were studied by Bruus et al. in Ref. 22.) Peaks corresponding to different wave functions are uncorrelated if measured at the same value of the magnetic field, but correlated at different values of the magnetic field. In order to describe these correlations, we still employ the Pandey-Mehta Hamiltonian (4) but now take \( \alpha \) and \( \alpha' \) large. The joint distribution of eigenvectors \( v_{\mu} \) and \( v_{\nu} \) at different values of \( \alpha \) is thus characterized by the unitary invariants

\[
\tilde{\rho}_{\mu\nu} = v_{\mu}^\dagger(\alpha)v_{\nu}(\alpha'), \tag{16}
\]

where \( v_{\mu}(\alpha) \) denotes the eigenvector of the \( \mu \)-th level at magnetic-field strength \( \alpha \). At fixed \( \tilde{\rho}_{\mu\nu} \), the eigenvector components are distributed according to a multivariate Gaussian distribution with covariance matrix determined by the pair correlator

\[
\langle [v_{\mu}\times(\alpha)v_{\nu}\times(\alpha')] \rangle_{\tilde{\rho}} = \frac{\delta_{\mu\nu}}{N}. \tag{17}
\]

The second moments \( \langle |\tilde{\rho}_{\mu\nu}|^2 \rangle \) are known in the regimes

\[
|\varepsilon_{\mu} - \varepsilon_{\nu}| \ll \Delta, |\alpha' - \alpha| \ll 1,
\]

\[
2(\alpha' - \alpha)^2 / (\alpha' - \alpha)^4 + 4 \pi^2 (\varepsilon_{\mu} - \varepsilon_{\nu})^2 / \Delta^2. \tag{18}
\]
spin-orbit scattering is described by the following two-dimensional effective Hamiltonian:

\[ H_{SO} = \frac{1}{2m} \left[ p + \sigma_1 \frac{\rho_2}{\lambda_2} - p \sigma_2 \frac{\rho_1}{\lambda_1} \right] \]

Here, \( \sigma_i \) are Pauli matrices, and \( \lambda_1, \lambda_2 \) are length scales associated with spin-orbit coupling along the directions \( \hat{x}_1 \) and \( \hat{x}_2 \) that span the plane in which the dot is formed. In the limit where \( \lambda_1, \lambda_2 \) are large compared to the linear dot size \( L \), the spin-orbit contribution can be mapped onto an effective magnetic field \( B_{SO} \) by means of a suitable unitary transformation of \( H_{SO} \):

\[ \tilde{H}_{SO} = \frac{1}{2m} (p - a_\perp)^2 - \frac{p^2}{2m} \]

is the vector potential that generates the leading spin-orbit effect. Hence, weak spin-orbit scattering takes the form of an effective magnetic field \( B_{SO} = \hbar c/2 e \lambda_1 \lambda_2 \) of opposite sign for the two spin directions and perpendicular to the plane of the two-dimensional electron gas in which the dot is formed. The parameter \( \alpha \) in the Pandey-Mehta Hamiltonian (4) is then given by

\[ \alpha = \pm \gamma \frac{V}{4\pi \lambda_1 \lambda_2} \sqrt{E_T / \Delta} \]

where the \( \pm \) corresponds to the two spin directions, and \( \gamma \) is the same geometric factor as in Sec. II. Experimental estimates suggest \( \gamma \approx 1 \), which implies that the effective magnetic field is weak enough to only partially break time-reversal symmetry.

The peak-height correlations for a weak magnetic field calculated in Sec. III A thus provide a good description of intrinsic peak-height correlations for a quantum dot with weak spin-orbit scattering in the absence of an external magnetic field. On the other hand, when a large external magnetic field \( B \) is applied perpendicular to the dot, electrons move in different effective magnetic fields \( B \pm B_{SO} \), depending on the direction of their spin. At zero temperature, conductance peaks correspond to resonant tunneling for one of the two spin directions. Our calculations in Sec. III B show that peaks originating from resonances with the same spin direction, i.e., \( |\alpha' - \alpha| = 0 \), will have uncorrelated heights, whereas peaks originating from resonances with opposite spin will have correlated heights, corresponding to the case of large magnetic fields with magnetic-field strength difference \( |\alpha' - \alpha| = \gamma (V/2 \pi \lambda_1 \lambda_2) \sqrt{E_T / \Delta} \).

“Scrambling” is the effect that each electron added to the quantum dot causes a small change to the self-consistent potential in the dot.\(^{26,27}\) Hence, every conductance peak is taken at a slightly different realization of the dot’s potential. While this leads to a decorrelation of peak heights corresponding to the same orbital state, scrambling also causes a positive correlation between peak heights corresponding to different or-
bital states, as we have shown above for the case of a large applied magnetic field. (In the unitary ensemble, a change in potential has the same effect as a change in the applied magnetic field. The situation at zero applied magnetic field would correspond to the orthogonal ensemble of random-matrix theory, for which the calculation proceeds along the same lines and gives similar results.) The effect of adding $n$ electrons to a disordered quantum dot corresponds to a parameter change $|\alpha' - \alpha| \sim n \sqrt{\Delta/E_T}$, where $E_T$ is the Thouless energy. Hence, we conclude from our calculations that the resulting correlations between peak heights are of order $n^2 \Delta/E_T$. While such correlations may be of numerical importance, their dependence on the ratio $E_T/\Delta$ is the same as that of the nonuniversal peak-height correlations in a disordered dot. We therefore see that both types of correlations need to be taken into account for a complete understanding of spectral scrambling effects.

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6. Thermal smearing, which causes several consecutive wave functions to contribute to a single conductance peak, becomes ineffective for temperatures $kT \ll \Delta$, although temperature effects remain relevant for $T$ as low as $k_B T \sim 0.1 \Delta$, see G. Usaj and H.U. Baranger, Phys. Rev. B 67, 121308 (2003); Y. Alhassid and T. Rupp, cond-mat/0212126 (unpublished).